Question: A fair die is rolled until it lands on a 6. A player will win the square, in dollars, of the number of rolls required. What is the expected win?

Scroll down one page for the answer.

Scroll down two pages for the solution.

Answer: $66

Solution

Let's find the general answer for a probability of p, which is 1/6 in this case.

The expected win is $p\,1^2 + p(1-p)2^2 + p(1-p)^2 3^2 + p(1-p)^3 4^2 + \ldots$

This can be expressed as:

$$\sum_{k=1}^{\infty} p(1-p)^{k-1}k^2$$

It's the k^2 term that makes this challenging.

Let's review the expected win if the player won the number of rolls required. That can be expressed as:

pr(get to first roll) + pr(get to second roll) + pr(get to third roll) + ...

= 1 + $(1-p)$ + $(1-p)^2$+ $(1-p)^3$+ $(1-p)^4$ + ... = $\frac{1}{p}$

That can be expressed as:

$\sum_{k=0}^{\infty}(1-p)^k = \frac{1}{p}$

Letting x=1-p:

$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Here is the humdinger! Take the derivative of both sides:

$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$

To clean up the left side, multiply both sides by x:

$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$

As a second humdinger, take the derivative again:

$\sum_{k=0}^{\infty} k^2 x^{k-1} = \frac{2x}{(1-x)^3} + \frac{1}{(1-x)^2}$

Let's again clean up the left side by multiplying by x:

$\sum_{k=0}^{\infty} k^2 x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}$

$= \frac{x(1+x)}{(1-x)^3}$

Scroll up to remind ourselves the task at hand is to solve for:

$$\sum_{k=1}^{\infty} p(1-p)^{k-1}k^2$$

This equals:

$\sum_{k=1}^{\infty} p(1-p)^k k^2 / (1\text{-}p) =$

$$\frac{p}{1-p}\sum_{k=1}^{\infty}(1-p)^k k^2$$

Let x = 1-p:

$$\frac{1-x}{x}\sum_{k=1}^{\infty} x^k k^2$$

$= \frac{1-x}{x} \times \frac{x(1+x)}{(1-x)^3}$

$$= \frac{(1+x)}{(1-x)^2}$$

In the original problem, p=1/6. Recall x = 1-p, so x=5/6.

Thus, the answer is:

$$\frac{(1+5/6)}{(1-5/6)^2} = \frac{(11/6)}{(1/6)^2} = \frac{396}{6} = 66$$

The general formula for probability p is:

$$\frac{2-p}{p^2}$$