

Question: What is the average number of people needed for there to be three people with the same birthday?

You may assume a 365-day year and every birthday has the same probability.

Go to page 2 for the answer.

Go to page 3 for the solution.

The answer is approximately 88.7389.

The key to solving this problem is that the answer is the same if we reword it as follows.

A random integer from 1 to 365 is chosen one at a time with an interval of time x between each one, where x follows an exponential distribution with a mean of 1. What is the average time required until a number is chosen three times?

Before getting started, recall that for any random variable the mean is the integral from 0 to infinity of the probability it hasn't happened. Let's take the case of x above.

$$\int_0^{\infty} e^{-t} dt =$$

$$-e^{-t} \text{ from 0 to infinity} =$$

$$-(0 - 1) = 1$$

The time between any given number being drawn is 365.

The Poisson distribution says the following are the probability an event has happened 0, 1 or 2 times in time t where the mean time between events is 365.

$$0 \text{ times: } \frac{(e^{-\frac{t}{365}})(\frac{t}{365})^0}{0!} = e^{-\frac{t}{365}}$$

$$1 \text{ time: } \frac{(e^{-\frac{t}{365}})(\frac{t}{365})^1}{1!} = e^{-\frac{t}{365}} \frac{t}{365}$$

$$2 \text{ times: } \frac{(e^{-\frac{t}{365}})(\frac{t}{365})^2}{2!} = e^{-\frac{t}{365}} \frac{(\frac{t}{365})^2}{2}$$

The probability an event has happened 0, 1 or 2 times is the sum of these three probabilities:

$$e^{-\frac{t}{365}} \left(1 + \frac{t}{365} + \frac{\left(\frac{t}{365}\right)^2}{2}\right)$$

The probability 365 different numbers have all been drawn 2 or less times in time t is the above probability to the power of 365:

$$\left(e^{-\frac{t}{365}} \left(1 + \frac{t}{365} + \frac{\left(\frac{t}{365}\right)^2}{2}\right)\right)^{365} =$$

$$e^{-t} \left(1 + \frac{t}{365} + \frac{\left(\frac{t}{365}\right)^2}{2}\right)^{365}$$

To get the mean time until the first number occurs three times is the integral of this from 0 to infinity:

$$\int_0^{\infty} e^{-t} \left(1 + \frac{t}{365} + \frac{\left(\frac{t}{365}\right)^2}{2}\right)^{365} dt$$

Solving that integral manually would certainly be a mess. It even timed out at my favorite integral calculator at www.integral-calculator.com. Fortunately, the integral calculator at wolframalpha.com was up to the challenge. Here is a direct link:

<https://www.wolframalpha.com/input?i=integral+zero+to+infinity+%28%28x+%2F+365%29%5E2+%2F+2+%2B+%28x+%2F+365%29+%2B+1%29%5E365+%2F+e%5Ex+dx>

The free version gives an answer to four decimal places as 88.7389.

To confirm this, I wrote a random simulation of over 2.3 billion trials. Here are the results:

PEOPLE	SIMULATION	PROBABILITY	PRODUCT
3	17,595	0.00000752577	0.000023
4	52,895	0.00002262435	0.000090
5	104,464	0.00004468155	0.000223
6	174,532	0.00007465117	0.000448
7	260,595	0.00011146221	0.000780
8	362,989	0.00015525837	0.001242
9	482,524	0.00020638612	0.001857
10	618,976	0.00026474964	0.002647
11	774,214	0.00033114834	0.003643
12	940,709	0.00040236192	0.004828
13	1,125,681	0.00048147851	0.006259
14	1,325,855	0.00056709733	0.007939
15	1,541,132	0.00065917603	0.009888
16	1,774,199	0.00075886391	0.012142
17	2,022,141	0.00086491415	0.014704
18	2,282,180	0.00097613855	0.017570
19	2,559,646	0.00109481685	0.020802
20	2,848,847	0.00121851448	0.024370
21	3,155,419	0.00134964206	0.028342
22	3,473,603	0.00148573635	0.032686
23	3,809,402	0.00162936496	0.037475
24	4,150,929	0.00177544357	0.042611
25	4,508,238	0.00192827248	0.048207
26	4,880,900	0.00208766821	0.054279
27	5,262,432	0.00225085783	0.060773
28	5,655,265	0.00241888114	0.067729
29	6,057,941	0.00259111452	0.075142
30	6,476,032	0.00276994123	0.083098
31	6,908,111	0.00295475091	0.091597
32	7,342,962	0.00314074624	0.100504
33	7,781,592	0.00332835793	0.109836
34	8,240,493	0.00352463998	0.119838
35	8,709,171	0.00372510386	0.130379
36	9,171,932	0.00392303691	0.141229
37	9,655,610	0.00412991662	0.152807
38	10,132,695	0.00433397636	0.164691
39	10,623,486	0.00454389846	0.177212
40	11,116,405	0.00475473075	0.190189
41	11,618,693	0.00496957037	0.203752
42	12,117,476	0.00518291082	0.217682
43	12,623,822	0.00539948614	0.232178
44	13,144,862	0.00562234640	0.247383

45	13,645,708	0.00583656924	0.262646
46	14,162,978	0.00605781699	0.278660
47	14,678,700	0.00627840262	0.295085
48	15,186,351	0.00649553611	0.311786
49	15,703,842	0.00671687839	0.329127
50	16,202,521	0.00693017436	0.346509
51	16,712,781	0.00714842377	0.364570
52	17,208,460	0.00736043658	0.382743
53	17,718,204	0.00757846529	0.401659
54	18,204,462	0.00778644852	0.420468
55	18,691,695	0.00799484879	0.439717
56	19,177,702	0.00820272466	0.459353
57	19,647,189	0.00840353457	0.479001
58	20,113,223	0.00860286756	0.498966
59	20,560,461	0.00879416108	0.518856
60	21,009,458	0.00898620697	0.539172
61	21,445,154	0.00917256372	0.559526
62	21,878,935	0.00935810139	0.580202
63	22,292,125	0.00953483183	0.600694
64	22,688,711	0.00970446038	0.621085
65	23,072,352	0.00986855207	0.641456
66	23,449,971	0.01003006802	0.661984
67	23,803,979	0.01018148502	0.682159
68	24,155,109	0.01033167104	0.702554
69	24,482,631	0.01047175940	0.722551
70	24,795,529	0.01060559275	0.742391
71	25,097,668	0.01073482424	0.762173
72	25,360,264	0.01084714232	0.780994
73	25,629,842	0.01096244676	0.800259
74	25,856,435	0.01105936556	0.818393
75	26,088,051	0.01115843282	0.836882
76	26,291,981	0.01124565816	0.854670
77	26,467,825	0.01132087050	0.871707
78	26,639,379	0.01139424792	0.888751
79	26,775,544	0.01145248868	0.904747
80	26,900,611	0.01150598259	0.920479
81	27,007,001	0.01155148793	0.935671
82	27,087,302	0.01158583443	0.950038
83	27,147,132	0.01161142504	0.963748
84	27,187,583	0.01162872682	0.976813
85	27,208,040	0.01163747673	0.989186
86	27,203,864	0.01163569056	1.000669
87	27,182,196	0.01162642268	1.011499

88	27,144,509	0.01161030312	1.021707
89	27,086,480	0.01158548285	1.031108
90	26,996,939	0.01154718419	1.039247
91	26,890,731	0.01150175670	1.046660
92	26,769,221	0.01144978420	1.053380
93	26,628,206	0.01138946898	1.059221
94	26,463,487	0.01131901505	1.063987
95	26,285,879	0.01124304820	1.068090
96	26,074,681	0.01115271417	1.070661
97	25,858,451	0.01106022785	1.072842
98	25,622,744	0.01095941079	1.074022
99	25,373,829	0.01085294437	1.074441
100	25,103,566	0.01073734694	1.073735
101	24,814,487	0.01061370150	1.071984
102	24,526,281	0.01049042946	1.070024
103	24,201,661	0.01035158235	1.066213
104	23,863,026	0.01020674072	1.061501
105	23,528,075	0.01006347481	1.056665
106	23,171,996	0.00991117200	1.050584
107	22,800,721	0.00975236952	1.043504
108	22,416,142	0.00958787663	1.035491
109	22,031,970	0.00942355782	1.027168
110	21,633,119	0.00925296048	1.017826
111	21,235,525	0.00908290079	1.008202
112	20,813,580	0.00890242564	0.997072
113	20,376,312	0.00871539650	0.984840
114	19,947,504	0.00853198590	0.972646
115	19,516,500	0.00834763601	0.959978
116	19,058,644	0.00815180094	0.945609
117	18,630,268	0.00796857511	0.932323
118	18,175,345	0.00777399454	0.917331
119	17,718,794	0.00757871765	0.901867
120	17,259,291	0.00738217811	0.885861
121	16,799,020	0.00718531009	0.869423
122	16,348,265	0.00699251227	0.853086
123	15,884,612	0.00679419769	0.835686
124	15,428,880	0.00659927110	0.818310
125	14,978,492	0.00640663025	0.800829
126	14,522,369	0.00621153642	0.782654
127	14,065,510	0.00601612778	0.764048
128	13,619,063	0.00582517259	0.745622
129	13,182,284	0.00563835261	0.727347
130	12,732,924	0.00544615146	0.708000

131	12,296,432	0.00525945424	0.688989
132	11,863,381	0.00507422881	0.669798
133	11,441,815	0.00489391576	0.650891
134	11,018,184	0.00471271947	0.631504
135	10,611,559	0.00453879702	0.612738
136	10,202,809	0.00436396566	0.593499
137	9,806,672	0.00419452916	0.574650
138	9,417,202	0.00402794428	0.555856
139	9,031,937	0.00386315797	0.536979
140	8,659,506	0.00370386105	0.518541
141	8,293,724	0.00354740805	0.500185
142	7,938,497	0.00339546965	0.482157
143	7,582,246	0.00324309327	0.463762
144	7,247,819	0.00310005149	0.446407
145	6,915,418	0.00295787628	0.428892
146	6,595,190	0.00282090772	0.411853
147	6,287,413	0.00268926473	0.395322
148	5,984,522	0.00255971160	0.378837
149	5,696,505	0.00243652040	0.363042
150	5,411,494	0.00231461493	0.347192
151	5,142,228	0.00219944394	0.332116
152	4,874,236	0.00208481787	0.316892
153	4,622,312	0.00197706444	0.302491
154	4,378,054	0.00187258992	0.288379
155	4,142,059	0.00177164967	0.274606
156	3,917,208	0.00167547596	0.261374
157	3,700,139	0.00158263078	0.248473
158	3,490,848	0.00149311242	0.235912
159	3,294,315	0.00140905093	0.224039
160	3,099,323	0.00132564857	0.212104
161	2,920,977	0.00124936606	0.201148
162	2,747,008	0.00117495570	0.190343
163	2,577,987	0.00110266170	0.179734
164	2,420,146	0.00103514964	0.169765
165	2,268,320	0.00097021032	0.160085
166	2,128,119	0.00091024327	0.151100
167	1,990,757	0.00085149052	0.142199
168	1,865,726	0.00079801202	0.134066
169	1,739,766	0.00074413616	0.125759
170	1,625,006	0.00069505078	0.118159
171	1,513,037	0.00064715918	0.110664
172	1,410,649	0.00060336558	0.103779
173	1,313,139	0.00056165841	0.097167

174	1,220,703	0.00052212150	0.090849
175	1,135,144	0.00048552604	0.084967
176	1,053,154	0.00045045711	0.079280
177	976,861	0.00041782492	0.073955
178	905,153	0.00038715383	0.068913
179	837,612	0.00035826506	0.064129
180	772,887	0.00033058076	0.059505
181	716,998	0.00030667580	0.055508
182	659,593	0.00028212242	0.051346
183	609,161	0.00026055155	0.047681
184	561,229	0.00024004998	0.044169
185	514,442	0.00022003815	0.040707
186	474,504	0.00020295579	0.037750
187	435,079	0.00018609285	0.034799
188	396,987	0.00016980007	0.031922
189	365,562	0.00015635890	0.029552
190	334,897	0.00014324281	0.027216
191	305,929	0.00013085256	0.024993
192	279,849	0.00011969757	0.022982
193	254,879	0.00010901735	0.021040
194	232,576	0.00009947787	0.019299
195	211,627	0.00009051752	0.017651
196	192,077	0.00008215555	0.016102
197	175,556	0.00007508916	0.014793
198	159,342	0.00006815408	0.013495
199	143,757	0.00006148803	0.012236
200	131,092	0.00005607093	0.011214
201	118,857	0.00005083775	0.010218
202	107,617	0.00004603016	0.009298
203	96,528	0.00004128715	0.008381
204	87,279	0.00003733115	0.007616
205	78,894	0.00003374470	0.006918
206	70,549	0.00003017536	0.006216
207	63,594	0.00002720055	0.005631
208	57,115	0.00002442934	0.005081
209	51,357	0.00002196652	0.004591
210	46,013	0.00001968077	0.004133
211	41,105	0.00001758151	0.003710
212	36,671	0.00001568499	0.003325
213	32,929	0.00001408446	0.003000
214	29,526	0.00001262892	0.002703
215	26,106	0.00001116611	0.002401
216	23,398	0.00001000784	0.002162

217	20,742	0.00000887181	0.001925
218	18,295	0.00000782517	0.001706
219	16,412	0.00000701977	0.001537
220	14,526	0.00000621309	0.001367
221	12,911	0.00000552232	0.001220
222	11,325	0.00000484395	0.001075
223	10,095	0.00000431785	0.000963
224	8,651	0.00000370022	0.000829
225	7,849	0.00000335719	0.000755
226	6,939	0.00000296796	0.000671
227	6,002	0.00000256719	0.000583
228	5,220	0.00000223271	0.000509
229	4,621	0.00000197650	0.000453
230	4,150	0.00000177505	0.000408
231	3,569	0.00000152654	0.000353
232	3,080	0.00000131738	0.000306
233	2,649	0.00000113304	0.000264
234	2,279	0.00000097478	0.000228
235	2,062	0.00000088196	0.000207
236	1,764	0.00000075450	0.000178
237	1,527	0.00000065313	0.000155
238	1,392	0.00000059539	0.000142
239	1,174	0.00000050215	0.000120
240	1,005	0.00000042986	0.000103
241	890	0.00000038067	0.000092
242	781	0.00000033405	0.000081
243	691	0.00000029556	0.000072
244	561	0.00000023995	0.000059
245	497	0.00000021258	0.000052
246	397	0.00000016981	0.000042
247	364	0.00000015569	0.000038
248	276	0.00000011805	0.000029
249	270	0.00000011548	0.000029
250	211	0.00000009025	0.000023
251	202	0.00000008640	0.000022
252	155	0.00000006630	0.000017
253	159	0.00000006801	0.000017
254	102	0.00000004363	0.000011
255	88	0.00000003764	0.000010
256	74	0.00000003165	0.000008
257	65	0.00000002780	0.000007
258	57	0.00000002438	0.000006
259	62	0.00000002652	0.000007

260	45	0.00000001925	0.000005
261	40	0.00000001711	0.000004
262	27	0.00000001155	0.000003
263	31	0.00000001326	0.000003
264	32	0.00000001369	0.000004
265	25	0.00000001069	0.000003
266	24	0.00000001027	0.000003
267	9	0.00000000385	0.000001
268	13	0.00000000556	0.000001
269	6	0.00000000257	0.000001
270	4	0.00000000171	0.000000
271	4	0.00000000171	0.000000
272	7	0.00000000299	0.000001
273	5	0.00000000214	0.000001
274	9	0.00000000385	0.000001
275	7	0.00000000299	0.000001
276	5	0.00000000214	0.000001
278	3	0.00000000128	0.000000
279	3	0.00000000128	0.000000
280	1	0.00000000043	0.000000
282	3	0.00000000128	0.000000
283	1	0.00000000043	0.000000
284	1	0.00000000043	0.000000
286	1	0.00000000043	0.000000
TOTAL	2,337,967,296	1.00000000000	88.739066

The simulation shows the median is 88 and the mode is 85.

I would like to thank Ace2 for asking this question and his help with the solution.