Question

A one-meter stick is broken in cut in two random places. What is the expected area of the smallest of the three pieces created?

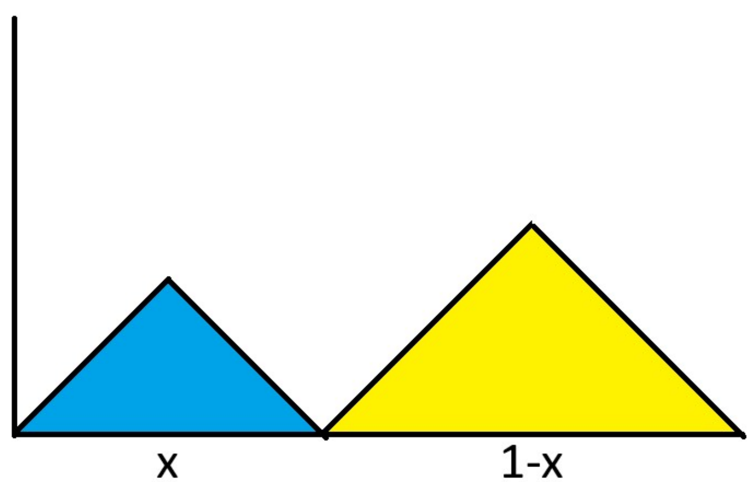
Answer

1/9 meter.

Solution

It does not change anything if we limit the first cut to somewhere between 0 and 0.5 on the stick.

First, let's examine the expected area of the smallest piece if the first cut is between 1/3 and ½. The following diagram shows the length of the smallest piece along the y-axis according to the second cut along the x-axis, where x is the location of the first cut.



The area of the blue triangle is (1/2) * (x/2) * x = $x^2/4$.

The area of the yellow triangle is $(x^2 - 2x + 1)/4$.

The sum of the two areas is $x^2/2 – x/2 + 1/4$.

To find the average area over this region, we integrate for x = 1/3 to 1/2:

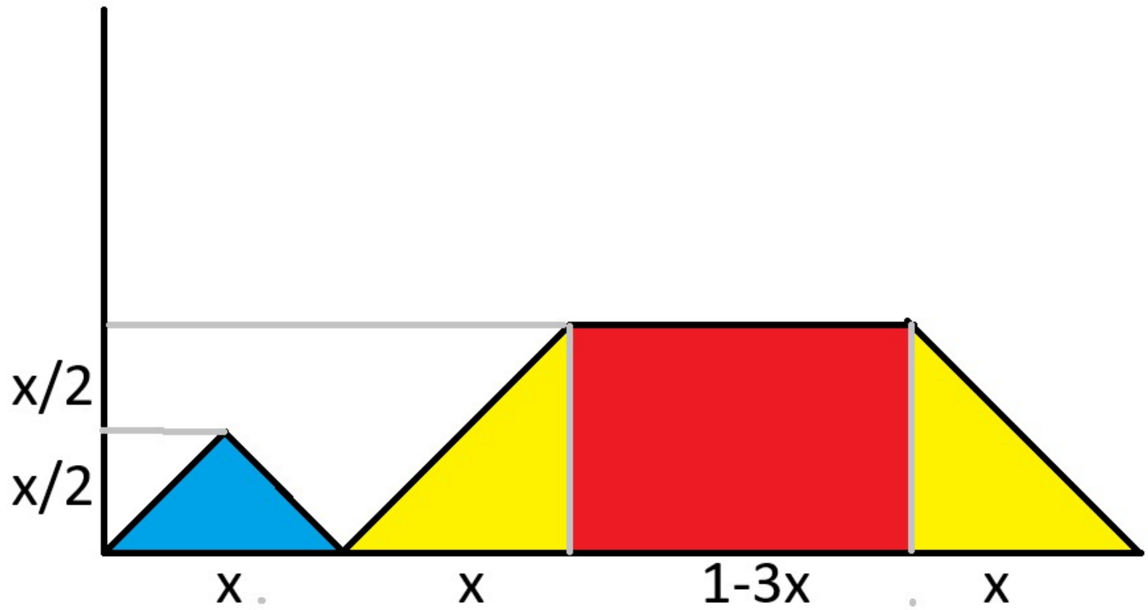
$$\int_{1/3}^{1/2} \frac{x^2}{2} - \frac{\text{x}}{2} + \frac{1}{4} dx =$$

$\frac{x^3}{6}$ - $\frac{x^2}{4}$ + $\frac{x}{6}$ from 1/3 to 1/2 =

= (1/48) – (1/6) + (1/8) – (1/162) + (1/36) – (1/2) = 161/7452 =~ 0.021605.

Let's not forget to multiply by 2 because we are only considering first cuts between 1/3 and 1/2. After adjusting for that, the average area is 161/3726 =~ 0.04321. Keep in mind this is the contribution to the answer if the first cut is between 1/3 and 2/3.

Next, let's consider what would happen if the first cut were between 0 and 1/3. The following diagram shows the length of the smallest piece along the y-axis according to the second cut along the x-axis, where x is the location of the first cut.



The area of the blue triangle is $x^2/2$.

The area of the combined yellow regions is x^2 .

The area of the red region is x - $3x^2$.

To find the average area, we integrate for x = 0 to 1/3.

$\int_0^{1/3} \frac{x^2}{2} + x^2 + \mathrm{x} - 3x^2 \; dx$ =

$\frac{x^2}{2} - \frac{x^3}{2}$ from 0 to 1/3 = (1/18) – (7/324) = 11/324.

Next, let’s double that to account for values of x from 2/3 to 1. So, the contribution to the answer if $x < 1/3$ or $x > 2/3$ is 22/324.

Finally add both pieces:

$161/3726 + 22/324 = 1/9$