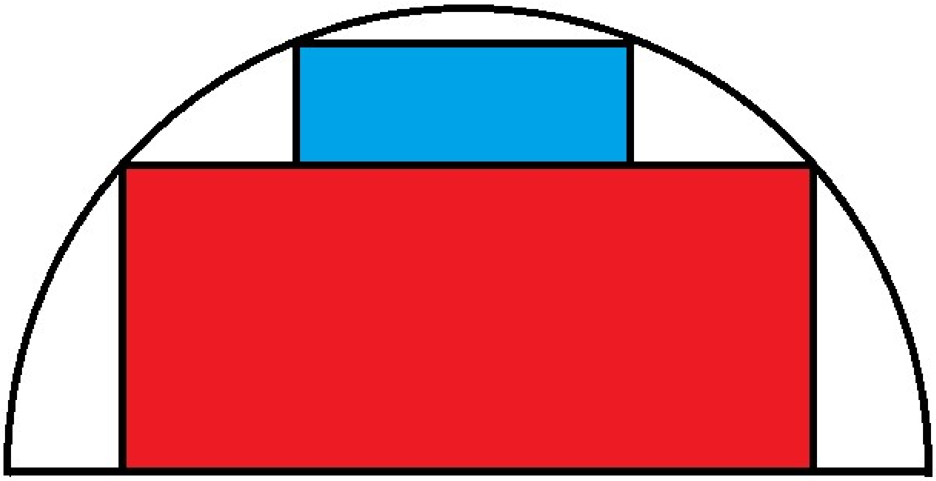
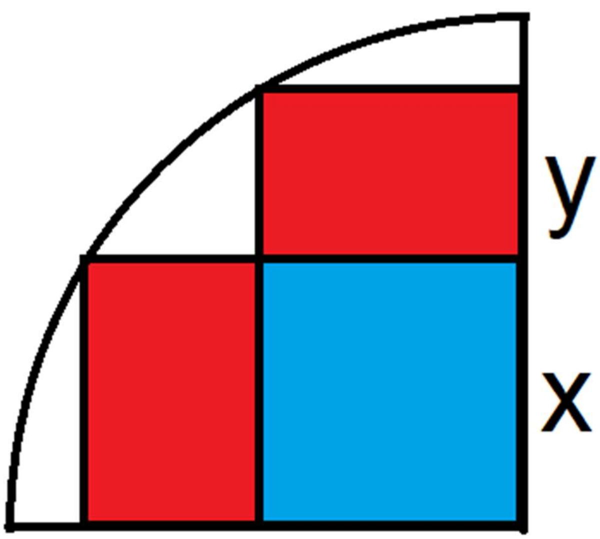
Q: What is the maximum combined area of the two inscribed rectangles in a unit semicircle?



Let's reword the problem to find inscribed rectangles in a quarter circle and then double the answer. Consider the image below, where the sides of the quarter circle on the bottom and right.



It stands to reason that in such a quarter circle, the answer would be the same whether:

a) The large rectangle is on the bottom and the small one on top.
b) The large rectangle is on the right and small one on the left.

In the preceding image, let the large rectangle be the blue region plus one red region. Likewise, the small rectangle is the other red region.

You can see the blue region is a square. Let:

x = side of the blue square

y = short side of red rectangles

By the Pythagorean formula:

$$x^2 + (x+y)^2 = 1$$

$$(x+y)^2 = 1 - x^2$$

$$x + y = \sqrt{1 - x^2}$$

$$y = \sqrt{1 - x^2} - x$$

The total area of the blue square and two red rectangles is:

$$x^2 + 2xy =$$

$$x^2 + 2x(\sqrt{1 - x^2} - x) =$$

$x^2 + 2x\sqrt{1 - x^2} - 2x^2$ =

f(x) = $2x\sqrt{1 - x^2}$ - x^2

To find the maximum area, take the derivative of the above function of the area and find the root.

f'(x) = $2x$ (1/2)(-$2x$)$(1 - x^2)^{-0.5} + 2(1 - x^2)^{+0.5}$ -$2x = 0$

$\frac{-2x^2}{\sqrt{1 - x^2}}$ + $2\sqrt{1 - x^2}$ = 2 x

$\frac{-2x^2}{\sqrt{1 - x^2}}$ + $\frac{2\,(1 - x^2)}{\sqrt{1 - x^2}}$ = 2 x

$-2x^2$ + 2 $(1 - x^2)$ = 2 $x\sqrt{1 - x^2}$

2 - $4x^2$= 2 $x\sqrt{1 - x^2}$

1 - $2x^2$= $x\sqrt{1 - x^2}$

$(1 - 2x^2)^2 = (x\sqrt{1 - x^2})^2$

$4x^4$ - $4x^2 + 1 = x^2\ (1 - x^2)$

$4x^4$ - 4 $x^2 + 1 = x^2 - x^4$

$5x^4$ - 5 $x^2 + 1 = 0$

Let $a = x^2$

$5a^2$ – 5 $a + 1 = 0$

By the Pythagorean formula:

$a = (5 \pm \sqrt{5}\ )\ /10$

Recall $x = \sqrt{a}$

x = $\sqrt{(5 \pm \sqrt{5}\ )\ /10}$ = 0.5257 or 0.8507

Recall that x = side of the square. The maximum x can be and still fit in the quarter circle is $\sqrt{2}/2$ =~ 0.7071. Thus, x must be the lower of the two values from the quadratic formula, of $\sqrt{\frac{5-\sqrt{5}}{10}}$ =~ 0.52573111211913

We already established that y = $\sqrt{1 - x^2} - x$. Let's solve for y.

$$y = \sqrt{1 - \left(\sqrt{\frac{(5-\sqrt{5}\ )}{10}}\right)^2} - \sqrt{\frac{5-\sqrt{5}}{10}} =$$

$$\sqrt{1 - \frac{(5-\sqrt{5}\ )}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}} =$$

$$\sqrt{\frac{10}{10} - \frac{(5-\sqrt{5}\ )}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}} =$$

$$\sqrt{\frac{5+\sqrt{5}}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}} =\sim 0.32491969623291$$

Recall the total area is x^2 + 2 x y =

$$\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)^2 + 2\sqrt{\frac{5-\sqrt{5}}{10}} * \left(\sqrt{\frac{5+\sqrt{5}}{10}} - \sqrt{\frac{5-\sqrt{5}}{10}}\right) =$$

$$\frac{5-\sqrt{5}}{10} + 2\sqrt{\frac{5-\sqrt{5}}{10}} * \sqrt{\frac{5+\sqrt{5}}{10}} - 2\sqrt{\frac{5-\sqrt{5}}{10}} * \sqrt{\frac{5-\sqrt{5}}{10}} =$$

$$\frac{5-\sqrt{5}}{10} + 2\sqrt{\frac{25-5}{100}} - 2\sqrt{(\frac{5-\sqrt{5}}{10})^2} =$$

$$\frac{5-\sqrt{5}}{10} + 2\sqrt{\frac{20}{100}} - 2\,\frac{5-\sqrt{5}}{10} =$$

$$\frac{-5+\sqrt{5}}{10} + 2\sqrt{\frac{2}{10}} =$$

$$\frac{-5+\sqrt{5}}{10} + 2\sqrt{\frac{2}{10}} =$$

$$\frac{\sqrt{5}-5}{10} + \frac{2\sqrt{10}\sqrt{2}}{10} =$$

$$\frac{\sqrt{5}-5}{10} + \frac{2\sqrt{20}}{10} =$$

$$\frac{\sqrt{5}-5}{10} + \frac{4\sqrt{5}}{10} =$$

$$\frac{5\sqrt{5}-5}{10} =$$

$$\frac{\sqrt{5}-1}{2} =\sim 0.618033988749895$$

Finally, remember this is the area for the inscribed rectangles under the quarter circle. The original problem asked about a semicircle. So double that area to

$\boldsymbol{\sqrt{5}-1}$ =~ 1.236067977

Here are the component parts of the original problem based on the semicircle.

Length of lower rectangle	1.701301616704080
Height of lower rectangle	0.525731112119134
Length of upper rectangle	1.051462224238270
Height of upper rectangle	0.324919696232906
Area of lower rectangle	0.894427190999916
Area of upper rectangle	0.341640786499874
Total area	1.236067977499790

Here are x and y, in terms of the earlier diagram of the quarter circle.

x =~ 0.52573111211913

y =~ 0.32491969623291