Question:

Three points are distributed randomly and uniformly along a circle with radius 1. What is the expected minimum distance between the three points, where distances are measured along the circumference?

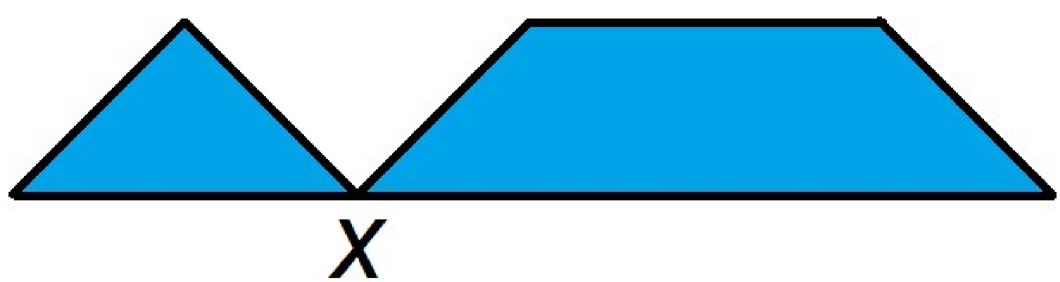
Solution:

For sake of visualization, let's cut the circle somewhere and lay it out flat. Then shrink it to a length of one. However, it will maintain it's circular property of a loop in that one end of the line will connect to the other.

Next, since the points can be anywhere on the circle, let's put one of them at the point where we cut the circle. On the line, that point will be on both the extreme left and right sides.

Let's let the second point be anywhere from the 0 to 1/2 point on the line. We can do this because the math would be the same if the second point were between 1/2 and 1. Let x be the distance from the second point to point 0.

One possibility is that x is between 0 and 1/3. Under that assumption, the following diagram would represent the shortest distance between two points on the line according to where the third point was in the range of 0 to 1.



The area of the blue region as a function of x is $x^2/4 + x/2 + x*(1-3x) + x/2 =$

$5x^2/4 + x - 3x^2 =$

$x - 7x^2/12$

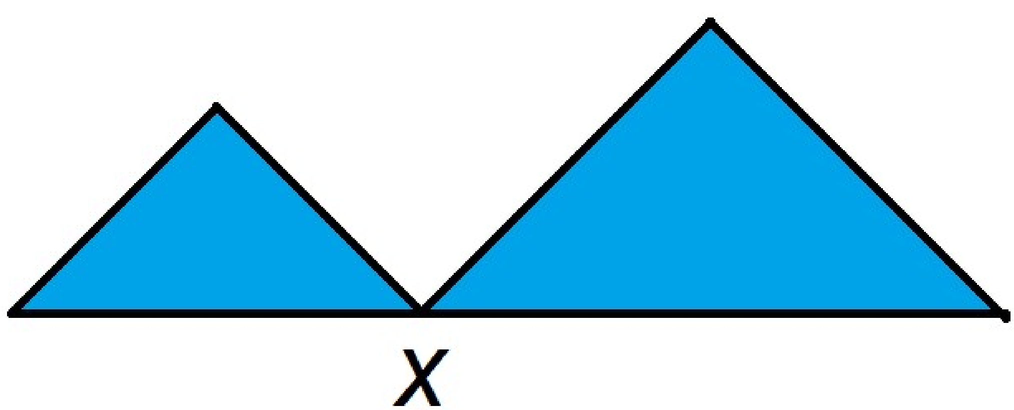
Let's find the average area of the blue region by integration from 0 to 1/3:

$$\int_0^{1/3} x - \frac{7x^2}{12} dx =$$

$\frac{x^2}{2} - \frac{7x^3}{36}$ from 0 to 1/3 =

$\frac{1}{18} - \frac{7}{324} = \frac{11}{324}$

Another possibility is x is between 1/3 and 1/2. The following diagram shows the distance to the nearest point according to values of x in that range.



The area of the blue region, as a function of x equals $x^2/4 + (1-x)^2/4 =$

$\frac{x^2}{4} + \frac{x^2 - 2\mathrm{x} + 1}{4}$

$= \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}$

Let’s find the average area of the blue region by integration from 1/3 to 1/2:

$$\int_{1/3}^{1/2} \frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} dx =$$

$\frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{4}$ from 1/3 to 1/2 =

1/48 – 1/16 + 1/8 – (1/162 – 1/36 + 1/12) =

7/324

Since the third point can fall anywhere between 0 and 1/2, we take the sum of the two integrals:

11/324 + 7/324 = 18/324 = 1/18

Next, double this, because the third point can also fall between 1/2 and 1:

2 * (1/18) = 2/18 = 1/9

Remember how we shrunk the circle down to a line of length one. The circumference of the circle is 2π. Now, expand the line back to the circumference of the circle by multiplying by 2π:

(1/9) * 2π = 2π/9 = apx. 0.698131701